530.327 - Introduction to Fluid Mechanics - Su
Applications of the momentum equation.

Reading: Text, §4.4 (The following examples are discussed in F.M. White, *Fluid Mechanics*, 5th ed.).

Example 1. Vector nature of Newton’s second law.

![Figure 1: A streamtube embedded in a steady flow.](image)

This example emphasizes that Newton’s second law deals with vector quantities. Figure 1 depicts a streamtube in a steady flow, where a streamtube is defined as follows—

- **Streamtube**: a closed surface whose sides are everywhere parallel to the local velocity vector.

For convenience, we will also say that the surfaces labeled 1 and 2 on the ends of the streamtube are perpendicular to the local velocity vector, and that density and velocity are uniform on those surfaces. What we want to know is, what force must act on the streamtube to cause the flow to bend as shown?

First, we will write the equation for conservation of mass for the system. (In general, even in problems where we’re concerned with forces and momenta, we need to start by ensuring that mass conservation is satisfied.) The general form of the conservation of mass is

\[
\frac{\partial}{\partial t} \int_{CV} \rho \, dV + \int_{CS} \rho (\vec{V} \cdot \vec{n} \, dA) = 0.
\]  

We will define the control volume here as being exactly the streamtube depicted in Fig. 1. In this case, the flow is steady, so the first term in the above equation is zero. Also, the only mass inflow/outflow terms on the control surface are on surfaces 1 and 2, because the velocity is always parallel to the sides of the streamtube. Thus, we have

\[
-\rho_1 V_1 A_1 + \rho_2 V_2 A_2 = 0
\]  

where \( V_1 \) and \( V_2 \) are the velocity magnitudes on surfaces 1 and 2. Equation (2) just expresses that for a streamtube in steady flow, what flows in one end has to flow out the other end. For convenience, we can define a mass flow rate for the system as

\[
\dot{m} = \rho_1 V_1 A_1 = \rho_2 V_2 A_2.
\]  

Now let’s look at the momentum equation, which in general form is

\[
\overrightarrow{F}_{CV} = \frac{\partial}{\partial t} \int_{CV} \rho \vec{V} \, dV + \int_{CS} \rho \vec{V} (\vec{V} \cdot \vec{n} \, dA).
\]  

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Of the terms on the right side, the first term is again zero because the flow is steady, and the second term is again evaluated on surfaces 1 and 2 only, since there is no transport of momentum across the walls of the streamtube. This gives us

\[
\vec{F}_{CV} = -\vec{V}_1 (\rho_1 V_1 A_1) + \vec{V}_2 (\rho_2 V_2 A_2) = \dot{m} (\vec{V}_2 - \vec{V}_1).
\]

This tells us the forces that need to be applied to the streamtube to get the velocity to change from \(\vec{V}_1\) at surface 1 to \(\vec{V}_2\) at surface 2. This change in velocity involves both the change in direction and the change in magnitude (for example, if the fluid is incompressible, then the flow decelerates through the streamtube as drawn – why?). This doesn’t say anything about what particular forces are applied (gravity, pressure, etc.), just what the net force needs to be.

**Example 2.** Forces required for flow deflection.

![Figure 2: Flow deflection by an angled vane.](image)

Figure 2 shows a jet of water deflected by a vane through an angle \(\theta\). We will assume that the flow is steady, that the water jet has uniform velocity across any cross-section, and that there are no frictional forces, so the velocity magnitude is constant and equal to \(V\). The vane is held fixed, the atmospheric pressure, \(p_0\), is constant, and we ignore gravity. What force is necessary to hold the vane in place?

Begin by defining the control surface to be the dotted surface shown in the figure. In selecting control surfaces, the goal is to simplify as much as possible the forces acting on the surface. For example, with the control volume as drawn in Fig. 2, the forces imposed by the vane on the fluid are collected in the force applied to the vane through the support member. (See also the text example 4.4 for a more thorough discussion of how to define control surfaces and volumes.)

The conservation of mass works out similarly to Example 1, except that we’re dealing specifically with water, which can be treated as having constant density, \(\rho\), and the velocity magnitudes are the same on surfaces 1 and 2 of the control volume, so

\[
-\rho \vec{V} A_{J1} + \rho \vec{V} A_{J2} = 0
\]

where \(A_{J1}\) and \(A_{J2}\) are the cross-sectional areas of the water jet only at surfaces 1 and 2 of the control volume. By (6), obviously \(A_{J1} = A_{J2}\), so the jet area doesn’t change as it’s deflected. We can then write a mass flow rate as

\[
\dot{m} = \rho V A.
\]
Now let’s look at the forces. We will make use of a common boundary condition on the pressure, namely

- The pressure in a free jet flow (i.e. a jet that is fully exposed to the atmosphere) will everywhere be approximately the same as the pressure of the atmosphere.

By this, we can neglect pressure forces, since they will balance around the control volume. The only force acting on the control volume is thus the force $\mathbf{F}$ applied through the vane support.

The momentum equation (4) then becomes

$$\mathbf{F} = \dot{m}(\mathbf{V}_2 - \mathbf{V}_1),$$

where $\mathbf{V}_2$ and $\mathbf{V}_1$ have the same magnitude, $V$, but different directions, as seen in Fig. 2. Since (8) is a vector equation, we can write it in terms of its directional components. Letting the $x$-direction be horizontal and the $y$-direction be vertical, we have

$$F_x = \dot{m}V(\cos \theta - 1)$$
$$F_y = \dot{m}V \sin \theta.$$  \hspace{1cm} (9)

We can also compute the magnitude of the force, as

$$F = (F_x^2 + F_y^2)^{1/2} = \dot{m}V(\sin^2 \theta + (\cos \theta - 1)^2)^{1/2} = \dot{m}V(2 - 2 \cos \theta)^{1/2} = 2\dot{m}V \sin \frac{\theta}{2}. \hspace{1cm} (10)$$

where the last step uses the half-angle formula for the sine function.

We can make some observations from (9) and (10).

- From (10), the maximum force, $F$, needed to hold the vane fixed occurs when the jet gets turned in the opposite direction, $\theta = 180^\circ$.

- We know that when $\theta$ is small, $\sin \theta \approx \theta$, and $\cos \theta \approx 1$. Then from (9), for small flow deflections $\theta$, there will be a force in the $y$-direction and essentially no horizontal force. How might this relate to the flow around an airplane wing?