Near the start of the course we pointed out that real-life data has certain properties, including the following:

- It is **discrete**, meaning that there are a finite number of values.
- It has **finite precision**, meaning that values are only known to a finite number of decimal places, in particular as measured by instruments and/or stored digitally. Computers, for example, store data values only to a finite number of bits.
- It is **uncertain**, which is the subject of these notes. It is worth noting that finite precision and uncertainty are often closely related, as will become clear.

The study of uncertainties is also known as error analysis. By ‘error’, it is important to note that we don’t mean ‘error’ in the context of ‘mistake’. Instead, ‘error’ or ‘uncertainty’ in the scientific sense means that any quantity that you measure and record will, most likely, differ, if only slightly, from the actual value of that quantity. This uncertainty can arise from a variety of sources; some examples are given below.

**Examples of measurement uncertainty.** For a simple illustration of measurement uncertainty, suppose that you have an analog clock with no second hand. At any instant, there is some actual time associated with the clock that goes out to an infinite number of decimal places. (For this discussion, it doesn’t matter whether the clock is ‘accurate’ in the sense of being consistent with the atomic clock standard or whatever, only that the clock is keeping time on some way.) However, you can only read the time off of the clock to some level of precision or certainty. For example, since the clock has no second hand, at any time you can only be sure of the time to within a minute, or maybe half a minute.

As another example, suppose you are measuring the length of your pen with a ruler. In this case, there are a variety of sources of uncertainty. The ruler might not be oriented exactly parallel to the pen; there is some uncertainty in the placement of the reference point, for example, you may be trying to put one end of your pen at the 1 cm point on the ruler, but you won’t be able to do this exactly accurately; and you will need to round up or down to the nearest ruler graduation.

When using a digital measurement device, we can be lulled into thinking that the numbers on the display are exact, but of course that isn’t true. In most cases, if you purchase a digital device, say a blood pressure reader, or a kitchen thermometer, or a battery tester, there will be a quoted estimate of the uncertainty accompanying any reading.

**Uncertainty analysis.** Uncertainties are usually quoted in the following format (for the case in which the quantity being measured is \( x \)):

\[
\text{Measured value} = x \pm \delta x, \quad \text{where} \quad \delta x > 0.
\]  

(1)

One can read this as meaning “I measure a value \( x \), but the actual value is likely to be up to \( \delta x \) away from that \( x \).” Here, \( \delta x \) represents a reasonable estimate of the uncertainty in the \( x \) measurement. You can, conceivably, quote \( \delta x \) as being some huge value, so that the true value of \( x \) will always be within the range given by Eq. 1, but the goal is to estimate the level of certainty by defining a range that just encompasses the likely values of \( x \). For example, using a metric ruler that has markings down to millimeters, the uncertainty in a length measurement is probably 1 mm. So, if I measure the diameter of a nickel as 21 mm, I would write that value as \( d = 21 \pm 1 \) mm.
There is a whole literature devoted to the quantification of uncertainties. You will almost certainly also cover or have covered this subject in physics lab. Here, we will focus on the propagation of uncertainties, that is, what happens when you compute a quantity using measured values that are individually subject to uncertainty. Two example problems that involve uncertainty propagation, which will we discuss in some detail, are:

1. Imagine that you are a flight dispatcher who works with cargo planes at an airport. In order to determine the fuel requirements for a given flight, you need to know the total weight of cargo loaded. To determine this, you weigh each item as it's loaded, using a scale that has a given measurement uncertainty that varies with each measurement.

2. You are a pavement engineer whose task is to measure the area of a rectangular lot in order to determine the amount of concrete you need to purchase. To compute the area, you measure the length of each side of the rectangle, using a surveying device with a given measurement uncertainty.

What we want to know now is, how to determine the uncertainty of a computation where the quantities used in the computation are themselves uncertain. We define the problem as follows. Suppose that the quantity we're interested in is $y$, and it's computed using $N$ different, independent, measured quantities, $x_i$, so

$$\text{Computed } y = f(x_1, x_2, \ldots, x_N) = y \pm \delta y. \quad (2)$$

Each $x_i$ has some associated uncertainty, so we can write

$$x_i = x_i \pm \delta x_i, \quad (3)$$

and our goal is to determine $\delta y$, the uncertainty in $y$ in Eq. 2, based on the values of $x_i$ and their uncertainties $\delta x_i$. We will assume that the uncertainties $\delta x_i$ are independent, that is, the instantaneous value of any one $\delta x_i$ has no effect on any others.

We will approach this problem in two steps. First, we need to know how the uncertainty in a single variable, $x_i$, affects the computed $y$. For this, we note that we can write

$$\frac{\Delta y}{\Delta x_i} = \frac{\text{change in } y}{\text{change in } x_i}, \quad (4)$$

that is, $\partial y/\partial x_i$ describes how $y$ changes as $x_i$ changes. Then, we can write

$$\left| \frac{\partial y}{\partial x_i} \right| \delta x_i = \delta y_i, \quad (5)$$

where $\delta y_i$ is the amount that $y$ changes when $x_i$ changes by $\delta x_i$. Since $\delta x_i$ is the uncertainty in the $x_i$ value, it represents the largest reasonable deviation of the true $x_i$ from the measured $x_i$. Thus the $\delta y_i$ in Eq. 5 is the largest reasonable deviation of the true $y$ from the computed value due to the uncertainty in $x_i$, i.e. it is the uncertainty in $y$ that arises due to the uncertainty in $x_i$.

Now, we want to know how the $\delta y_i$, the uncertainties in $y$ due to the the different $\delta x_i$, combine. One possibility is that they just add, as shown in Fig. 1a for the case of two variables $x_1$ and $x_2$. Then, we write

$$\delta y = \delta y_1 + \delta y_2. \quad (6)$$

However, a simple argument shows why this isn’t appropriate. The quantity $\delta y$ is supposed to represent a reasonable deviation of the computed $y$ from the actual $y$. As given by Eq. 6, though, $\delta y$ results from $x_1$ and $x_2$ being simultaneously at their maximum reasonable deviations from their
measured values. The problem with this is that the uncertainties in \(x_1\) and \(x_2\) are independent, so in fact it is not reasonable to expect that \(x_1\) and \(x_2\) will simultaneously be at their maximum \(\delta x_1\) and \(\delta x_2\). Thus, the \(\delta y\) given by Eq. 6 overestimates the true uncertainty of \(y\).

The appropriate way to combine the individual uncertainties \(\delta y_i\) is to recognize that since they’re independent, their lengths add up like the perpendicular sides of a right triangle, as shown in Fig. 1b. Thus we have, for two measured variables,\
\[
\delta y = (\delta y_1^2 + \delta y_2^2)^{1/2}. \tag{7}
\]
For the general case of \(N\) different measured variables, the relevant formula is\
\[
\delta y = \left(\sum_{i=1}^{N} \delta y_i^2\right)^{1/2} = \left[\sum_{i=1}^{N} \left(\frac{\partial y}{\partial x_i} \delta x_i\right)^2\right]^{1/2}. \tag{8}
\]
Armed with this formula, let’s go back and look at the example problems mentioned above.

**Example 1.** Let there be \(N\) items loaded onto the plane, with the weight measurement for each item being given by \(x_i = x_i + \delta x_i\). The total weight loaded is given by\
\[
y = \sum_{i=1}^{N} x_i. \tag{9}
\]
To compute the uncertainty in \(y\), we recognize that \(\partial y/\partial x_i = 1\) for all \(i\), so \(\delta y_i = \delta x_i\). Then, by Eq. 8, the uncertainty in \(y\) is given by\
\[
\delta y = \left(\sum_{i=1}^{N} \delta x_i^2\right)^{1/2}. \tag{10}
\]
Now let’s look at some numbers. Suppose that there are three heavy items loaded onto the plane, whose weights are: 15,210 ± 300 lbs; 24,984 ± 600 lbs; and 12,140 ± 300 lbs. By Eqs. 9 and 12, the total weight is given by\
\[
52,334 \pm 735 \text{ lbs.} \tag{11}
\]
From the standpoint of the cargo operator, you can imagine that the goal is to get an upper bound on the total weight, since it is desirable to load only as much fuel as might be necessary to make a given trip. Based on this result of Eq. 11, one can say that the total cargo weight is likely to be less than 53,069 lbs.
Example 2. In this case, \( y \) is the area of a rectangular lot and \( x_1 \) and \( x_2 \) are the side lengths, so

\[
y = x_1 \cdot x_2.
\] (12)

In this case, the \( \delta y_i \) are given by

\[
\delta y_1 = \left| \frac{\partial y}{\partial x_1} \right| \delta x_1 = |x_2| \delta x_1
\]

\[
\delta y_2 = \left| \frac{\partial y}{\partial x_2} \right| \delta x_2 = |x_1| \delta x_2,
\] (13)

so by Eq. 8, \( \delta y \) is given by

\[
\delta y = \left( x_2^2 \delta x_1^2 + x_1^2 \delta x_2^2 \right)^{1/2}.
\] (14)

Suppose that the measured side lengths are 180 ± 2 ft and 110 ± 2 ft. By Eqs. 12 and 14, the total area can be written

\[
19,800 \pm 422 \text{ ft}^2.
\] (15)

The pavement engineer is interested in the highest possible area given the uncertainty in his measurements, in order to ensure that he purchases enough concrete to finish the job. In this case, the area of the rectangular lot is likely to be less than 20,222 ft\(^2\).